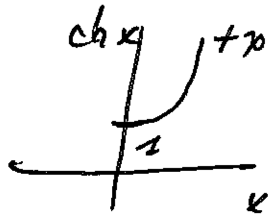
Exo $f(x,y) = (x^2+2x+2)\cosh(x+y+3)$

$f : \mathbb{R}^2 \to \mathbb{R}$

Rappel $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$. Fct paire de classe $\mathcal{C}^\infty$ sur $\mathbb{R}$

$\cosh'(z) = \sinh'(z)$



Notations : ch et sh sont ok.

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f de classe $\mathcal{C}^2$ (même $\mathcal{C}^\infty$) sur $\mathbb{R}^2$

- Pt(s) critique(s).

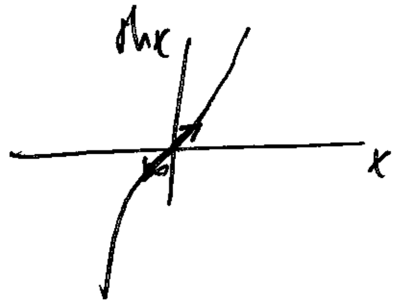
$$\partial_x f(x,y) = (2x+2)\,\mathrm{ch}(x+y+3) + (x^2+2x+2)\,\mathrm{sh}(-)$$
$$\partial_y f(x,y) = \mathrm{sh}(x+y+3).(x^2+2x+2)$$

$$\nabla f(x,y) = 0 \iff \begin{cases} (2x+2)\,\mathrm{ch}(x+y+3) = -(x^2+2x+2)\,\mathrm{sh}(-) \\ x+y+3 = 0 \text{ ou } (x+1)^2 + 1 = 0 \end{cases}$$

Impossible.

Rappel $\mathrm{sh}(x) = \frac{1}{2}(e^z - e^{-z})$. Fct impaire de classe $\mathcal{C}^\infty$ sur $\mathbb{R}$.

$\mathrm{sh}'x = \mathrm{ch}\,x$



sh x strict croissante sur $\mathbb{R}$

$$\nabla f(x,y) = 0 \iff \begin{cases} 2x+2 = 0 \\ x+y+3 = 0 \end{cases} \iff \begin{cases} x = -1 \\ y = -3+1 = -2 \end{cases}$$

f admet un unique pt critique sur $\mathbb{R}^2$

$$(x_0, y_0) = (-1, 2).$$

- Calcul de $H_f(x_0, y_0)$.

$$\partial^2_{xx} f(x,y) = 2\,ch(z) + (2x+1)\,sh(z) + (2x+2)\,sh(z) + (x^2+2x+2)\,ch(z)$$

avec $z = x+y+3$.

$$\Rightarrow \partial^2_{xx} f(x_0, y_0) = 2 + (1 - 2 + 2) = +3$$

- $\partial^2_{xy} f(x,y) = (x^2+2x+2)\,ch(z) + sh(z)(2x+1) \Rightarrow \partial^2_{xy} f(x_0,y_0) = 1$
- $\partial^2_{yy} f(x,y) = (x^2+2x+2)\,ch(z) \Rightarrow \partial^2_{yy} f(x_0, y_0) = 1$

  avec $z = x+y+3$

Soit

$$H_f(x_0, y_0) = \begin{pmatrix} +3 & 1 \\ 1 & 1 \end{pmatrix}$$

$\det(H_f(x_0,y_0)) = 3 - 1 = 2$

$tr(H_f(x_0,y_0)) = 4 > 0$

les 2 v.p. st tq : $\lambda_1 > 0$, $\lambda_2 > 0$

Dc f admet un unique min. local ds $\mathbb{R}^2$.

Ce min local est-il un min. global ?

On a : $f(x_0, y_0) = \operatorname{ch}(0) = 1$

Or $f(x,y) = [\underbrace{(x+1)^2}_{\geq 0} + 1] \underbrace{\operatorname{ch}(x+y+3)}_{\geq 1}$

$\Rightarrow f(x,y) \geq 1 \quad \forall (x,y) \in \mathbb{R}^2.$

~~Dc le min local~~

Dc f atteint ~~un~~ un min. global au pt $(-1,-2)$.

Exo 2 $\quad f : B(0,r) \longrightarrow \mathbb{R}$

$$f(x,y,z) = x^3 + y^3 + z^3 - 3z - 3y$$

f est de classe $\mathcal{C}^\infty$ sur $B(0,r)$.

- Pts critiques.

$$\partial_x f(x,y,z) = 3x^2$$
$$\partial_y f(x,y,z) = 3y^2 - 3$$
$$\partial_z f(x,y,z) = 3z^2 - 3$$

$\nabla f(x,y,z) = 0 \iff \begin{cases} x = 0 \\ y = \pm 1 \\ z = \pm 1 \end{cases}$

Dc f admet 4 pts critiques :

$$\underset{P_0}{\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}} \underset{P_1}{\begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix}} \underset{P_2}{\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}} \underset{P_3}{\begin{pmatrix} 0 \\ +1 \\ +1 \end{pmatrix}}$$

- Calculons $H_f(x,y,z)$

$\partial^2_{xx} f(x,y,z) = 6x \;;\; \partial^2_{xy} f(x,y,z) = 0 = \partial^2_{xz} f(-)$
$\forall (x,y,z) \in \mathbb{R}^3$.

$\Rightarrow \partial^2_{xx} f(P_i) = 0 \quad \forall i = 1..4$

$\partial^2_{yz} f(x,y,z) = 0 \;;\; \partial^2_{yy} f(x,y,z) = 6y$

$\partial^2_{zz} f(x,y,z) = 6z$

D'où les 4 Hessiennes :

$$H_f(P_i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pm 6 & 0 \\ 0 & 0 & \pm 6 \end{pmatrix}$$

D'où $\det(H_f(P_i)) = 0 \quad \forall i = 1..4$

Dc aucun extremum local.