Exo

1) $f : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (x^3 y, y^2 e^{-xy})$

Matrice Jacobienne. f $\mathcal{C}^1$ sur $\mathbb{R}^2$. $\forall (x,y) \in \mathbb{R}^2$,

$$J_f(x,y) = \begin{pmatrix} 3x^2 y & x^3 \\ -y^3 e^{-xy} & e^{-xy}(2y - xy^2) \end{pmatrix}$$

2) $g : \mathbb{R}^3 \to \mathbb{R}$

g est de classe $\mathcal{C}^1$ sur $\Omega = \{(x,y,z) \in \mathbb{R}^3 \text{ tq} : (x,y) \neq (0,0)\}$

Vecteur gradient de g pour $\forall (x,y,z) \in \Omega$:

$$\nabla g(x,y,z) = (\partial_x g(x,y,z), \partial_y g(x,y,z), \partial_z g(x,y,z))^T$$

$$\partial_x g(x,y,z) = \frac{1}{(x^2+y^2)}\left(z - \frac{2x^2 z}{(x^2+y^2)}\right)$$

$$\partial_y g(x,y,z) = -2\frac{xyz}{(x^2+y^2)^2}$$

$$\partial_z g(x,y,z) = \frac{x}{(x^2+y^2)}$$

Exo Laplacien

$k \in \mathbb{N}_+^*$ $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \ln\left((x^2-y^2)^k\right)$$

1) f de classe $\mathcal{C}^2$ sur $\Omega = \{(x,y) \in \mathbb{R}^2 \text{ tq } x^2 > y^2\}$

2) $\forall (x,y) \in \Omega$,

$(\ln u)' = \frac{u'}{u}$

$$\partial_x f(x,y) = k(x^2-y^2)^{k-1}\, 2x \,\frac{1}{(x^2-y^2)^k} = \frac{2kx}{(x^2-y^2)}$$

$$\partial_y f(x,y) = \frac{-2ky}{(x^2-y^2)}$$

$$\partial^2_{xx} f(x,y) = -\frac{2k}{(x^2-y^2)}\left(-1 + \frac{2x^2}{(x^2-y^2)}\right)$$

$$\partial^2_{yy} f(x,y) = \frac{-2k}{(x^2-y^2)}\left(1 + \frac{2y^2}{(x^2-y^2)}\right)$$

D'où $\forall (x,y) \in \Omega$,

$$\Delta f(x,y) = \frac{-4k}{(x^2-y^2)^2}(x^2+y^2)$$

3) D'où

$\Delta f(x,y) = 0 \ \forall (x,y) \in \Omega$

$\Leftrightarrow$ $k = 0$

or $k \geqslant 1$. Dc pas de sol°.

Exo Dérivat° fcts composées

$f : \mathbb{R}^2 \to \mathbb{R}$ de classe $\mathcal{C}^1$

$g(x, y) = f(x^2 - y^2, 2xy)$.

$g : \mathbb{R}^2 \to \mathbb{R}$. g de classe $\mathcal{C}^1$ sur $\mathbb{R}^2$.

$$\partial_x g(x,y) = \partial_x f(x^2-y^2, 2xy) \cdot 2x + \partial_y f(x^2-y^2, 2xy) \cdot 2y$$

$$\partial_y g(x,y) = \partial_x f(x^2-y^2, 2xy) \cdot (-2y) + \partial_y f(x^2-y^2, 2xy) \cdot 2x$$
$$= 2\left(-y\, \partial_x f(-) + x\, \partial_y f(-)\right)$$

Exo Extrema

$$f : \Omega \to \mathbb{R} \qquad \Omega =]-\pi ; +\pi[\times \mathbb{R}$$

$$f(x,y) = \sin x + y^2 - 2y + 1$$

1. f est la Σ de fcts $\mathcal{C}^\infty$ sur $\mathbb{R}^2$.

2. $\partial_x f(x,y) = \cos x \quad ; \quad \partial_y f(x,y) = 2(y-1)$

$$\nabla f(x,y) = (\cos x, 2(y-1))^T. \qquad \forall (x,y) \in \Omega.$$

$$\partial^2_{xx} f(x,y) = -\sin x \quad ; \quad \partial^2_{xy} f(x,y) = 0$$

$$\partial^2_{yy} f(x,y) = 2$$

$$\forall (x,y) \in \Omega, \quad H_f(x,y) = \begin{pmatrix} -\sin x & 0 \\ 0 & +2 \end{pmatrix}$$

3. Pts critiques. $\nabla f(x,y) = 0 \iff \begin{cases} x = \pm \frac{\pi}{2} \\ y = 1 \end{cases}$

Notons : $P_- \begin{pmatrix} -\pi/2 \\ +1 \end{pmatrix} \qquad P_+ \begin{pmatrix} +\pi/2 \\ +1 \end{pmatrix}$

$$H_f(P_-) = \begin{pmatrix} +1 & 0 \\ 0 & +2 \end{pmatrix} \qquad H_f(P_+) = \begin{pmatrix} -1 & 0 \\ 0 & +2 \end{pmatrix}$$

v.p. : $+1, +2$ — P_- min. local

v.p. : $-1, +2$ — P_+ pt selle

4. On a : $f(P_-) = -1$

Or $f(x,y) = \underbrace{\sin x}_{\geq -1} + \underbrace{(y-1)^2}_{\geq 0} \geq -1$ sur Ω.

Donc P_- est un min. global de f sur Ω.

Exo Dérivée directionnelle.

$$f(x,y) = x^2(y+1) \qquad a = (2,1) \qquad d = (1,1)$$

1) Def$^{\circ}$.

$$D_d f(a) = \frac{1}{t} \lim_{t \to 0} \big(f(a+td) - f(a) \big)$$

$$f(a+td) = f(2+t,\, 1+t) = (2+t)^2(2+t) = (2+t)^3$$

$$f(a) = f(2,1) = 4 \cdot 2 = 8$$

$$\frac{1}{t}\big(f(a+td) - f(a)\big) = \frac{1}{t}\big((2+t)(4+4t+t^2) - 8\big)$$

$$= \frac{1}{t}\big(\cancel{8} + 12t + 6t^2 + t^3 \cancel{- 8}\big)$$

$$= 12 + 6t + t^2 \xrightarrow[t \to 0]{} \underline{12}$$

2) On a : $D_d f(a) = Df(a)(d) = \langle \nabla f(a), d \rangle$

$$\partial_x f(x,y) = 2x(y+1) \; ; \; \partial_x f(a) = 8$$

$$\partial_y f(x,y) = x^2 \; ; \; \partial_y f(a) = 4$$

$$D_d f(a) = 8 \cdot 1 + 4 \cdot 1 = \underline{12}$$