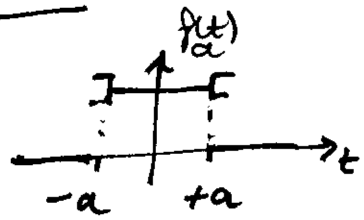
Exo — Fct porte et $\int \frac{\sin u}{u} du$

On pose :

1. $f_a(t) = \begin{cases} 1 & \text{pour } |t| < a \\ 0 & \text{pour } |t| \geq a \end{cases}$



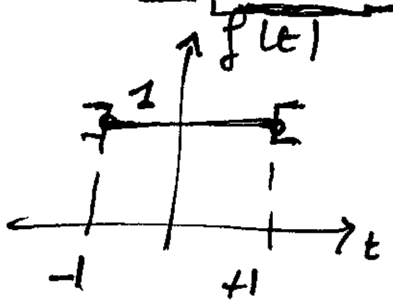
$$\hat{f}(k) = \int_{-a}^{+a} e^{-ikx}\, dx = 2\int_0^a \cos(kx)\, dx - 0$$

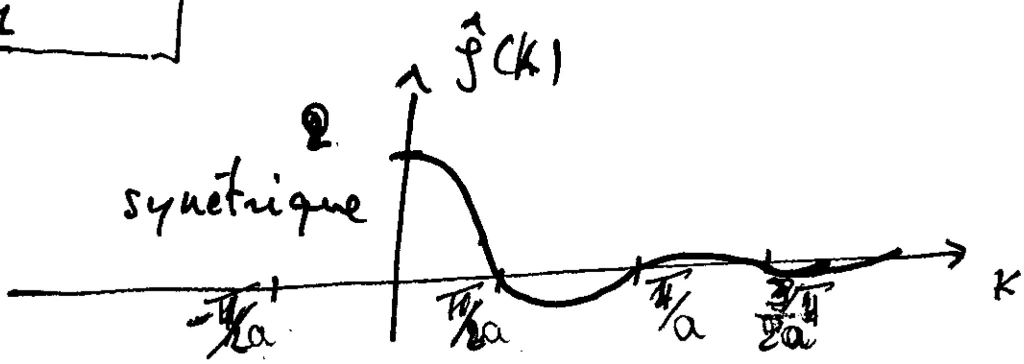
$$= 2\,\frac{1}{k}\left[\sin(kx)\right]_0^a \quad k \neq 0$$

$$= \frac{2\sin(ka)}{k} \quad \text{pour } k \neq 0$$

Et pour $k=0$, $\hat{f}(k) = 2a$.

2. Traçons ds le cas $a = 1$





3. Egalité de Parseval. Prenons $a = 1$.

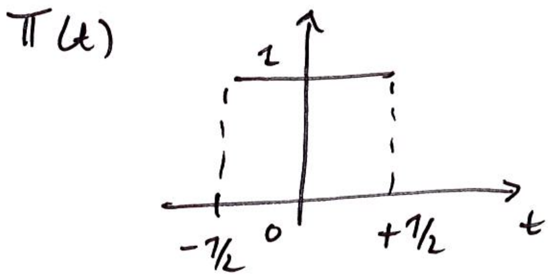
$$\underbrace{\int_{-\infty}^{+\infty} |f(x)|^2\, dx}_{} = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{+\infty} |\hat{f}(k)|^2\, dk}_{}$$

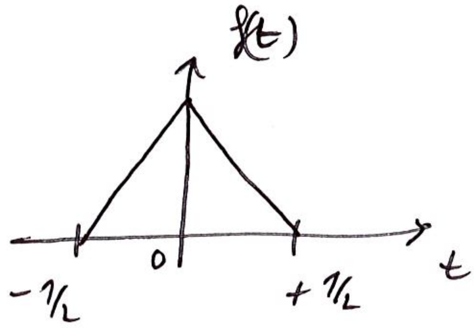
$$= \int_{-1}^{+1} 1\, dx \qquad\qquad a=1 \;\; = \frac{1}{2\pi}\int_0^{+\infty} \frac{8\sin^2(k)}{k^2}\, dk$$

$$= 2$$

D'où $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2}\, du = \pi$ (car $\frac{\sin^2 u}{u^2}$ fct paire)

Exo. T.F. d'un signal élémentaire





signal "triangle"

Rem : f est paire.

Comme $f(t)$ est paire, on a :

$$\hat{f}(k) = 2\int_0^{1/2} \underbrace{(1-2x)}_{v}\ \underbrace{\cos(kx)}_{u'}\,dx$$

$$\underset{IPP}{=} \underbrace{\left[\frac{1}{k}\sin(kx)\cdot(1-2x)\right]_0^{1/2}}_{=0} - \frac{2}{k}\int_0^{1/2}(-2)\cdot\sin(kx)\,dx \quad \text{pour } k \neq 0$$

$$= +\frac{4}{k}\left[\frac{-\cos(kx)}{k}\right]_0^{1/2}$$

$$= \frac{4}{k^2}\left(-\cos\left(\frac{k}{2}\right)+1\right).$$

Or : $\forall a,\ 1-\cos(2a) = 2\sin^2 a$ (f^{ule} de trigo.)

D'où :

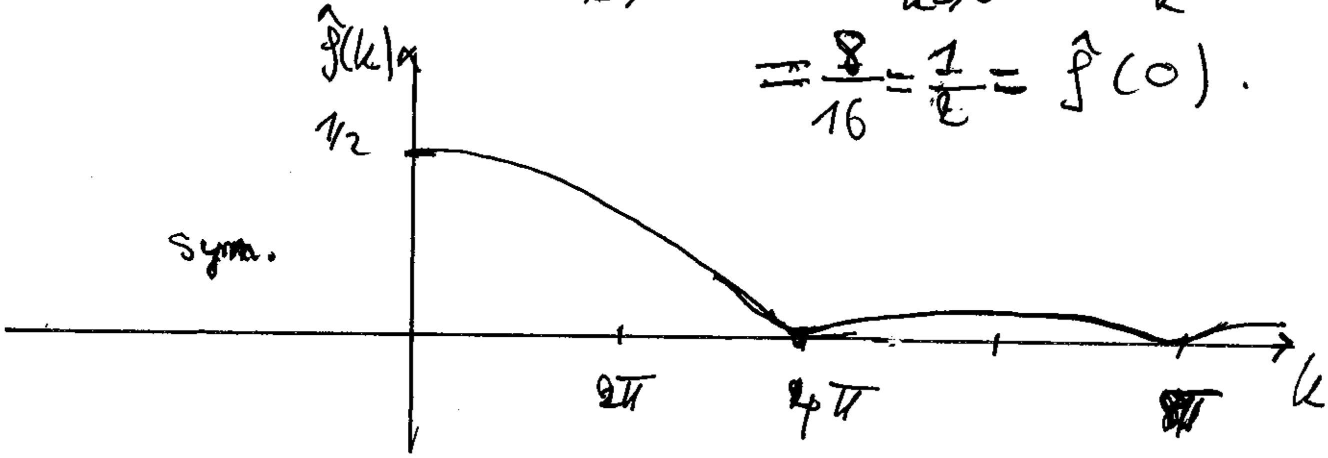
$$\forall k \neq 0,\ \hat{f}(k) = 8\,\frac{\sin^2\left(\frac{k}{4}\right)}{k^2} \quad \text{et} \quad \hat{f}(0) = 2\int_0^{1/2}(1-2x)\,dx$$

$$\Rightarrow \hat{f}(0) = \frac{1}{2}$$

Graphe de $\hat{f}(k)$

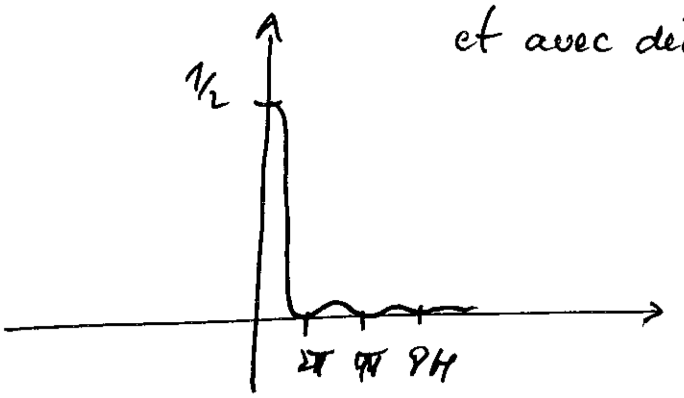
Remarque(*) : on a $\lim_{k\to 0} \hat{f}(k) = \lim_{k \to 0} \frac{8 \sin^2(\frac{k}{4})}{k^2}$

$$= \frac{8}{16} = \frac{1}{2} = \hat{f}(0).$$



$\hat{f}(k)$ fct paire, positive, qui s'annulle pour : $k = 2\pi m$; $m = 1, 2, \ldots$

et avec décroissance à l'∞ en $\frac{1}{k^2}$.



(*) Rappel. $\sin u \underset{0}{\sim} u$ dc $\sin^2 u \underset{0}{\sim} u^2$

et $\frac{\sin^2 u}{u^2} \underset{0}{\sim} 1$.

Exo. TF & valeurs d'intégrales

$$\Pi_T(x) = \begin{cases} 1 & \text{pour } x \in [-T/2, T/2] \\ 0 & \text{sinon} \end{cases}$$

On a :

$$\mathcal{F}(\Pi_T(x))(k) = \frac{2}{k} \sin\left(\frac{T}{2} k\right) \text{ pour } k \neq 0$$

$$= T \quad \text{pour } k = 0$$

Par ailleurs,

$$\Pi_T(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\Pi_T}(k)\, e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2}{k} \sin\left(\frac{T}{2} k\right) \cos(kx)\, dk + \frac{i}{2\pi} \int_{\mathbb{R}} \frac{2}{k} \sin\left(\frac{T}{2} k\right) \sin(kx)\, dx$$

(second terme barré) fct impaire $\Rightarrow \int_{\mathbb{R}} - \, dk = 0$

On a donc :

$$\boxed{I(x) = \int_{\mathbb{R}} \sin\left(u \frac{T}{2}\right) \cos(ux) \frac{du}{u} = \pi \cdot \Pi_T(x)}$$

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On pose $x = 0$, il vient :

$$I(0) = \int_{\mathbb{R}} \frac{1}{u} \sin\left(u \frac{T}{2}\right) du = \pi .$$

chgt de var. $z = u \frac{T}{2} \Rightarrow dz = \frac{T}{2} du$

$\Rightarrow u = \frac{2z}{T}$

D'où :

$$\int_{-\infty}^{+\infty} \frac{T}{2} \cdot \frac{1}{z} \cdot \sin z \cdot \frac{2}{T} \, dz = \pi$$

$$\Rightarrow \boxed{\int_0^{+\infty} \frac{\sin z}{z} \, dz = 2\pi} \quad \left(\text{car } \frac{\sin z}{z} \text{ paire}\right)$$